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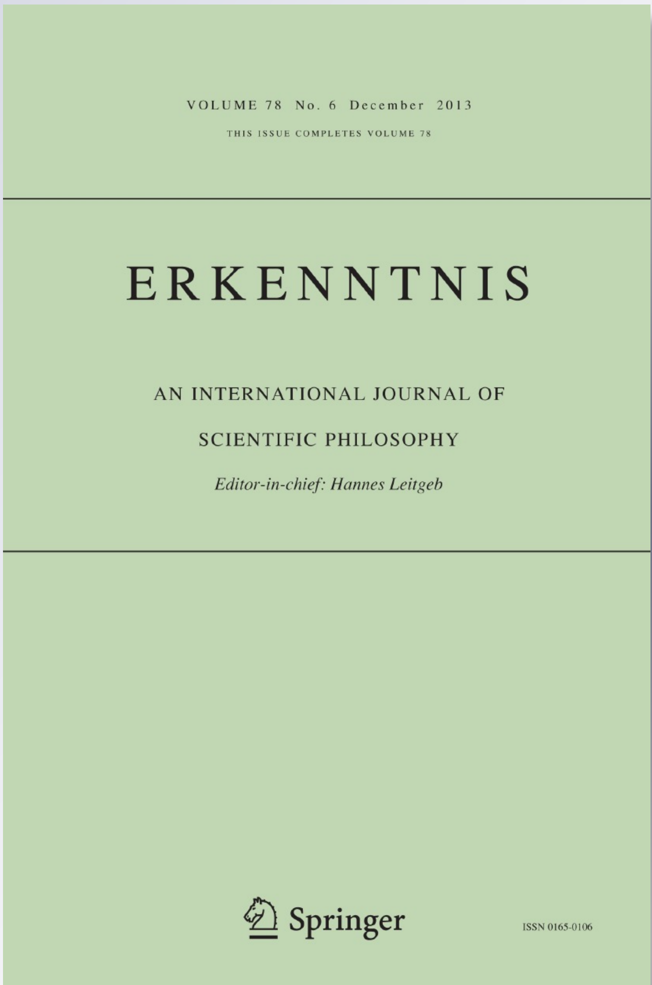

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ORIGINAL ARTICLE

The Forgotten Tradition: How the Logical Empiricists Missed the Philosophical Significance of the Work of Riemann, Christoffel and Ricci

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Abstract This paper attempts to show how the logical empiricists' interpretation of the relation between geometry and reality emerges from a "collision" of mathematical traditions. Considering Riemann's work as the initiator of a 19th century *geometrical* tradition, whose main protagonists were Helmholtz and Poincaré, the logical empiricists neglected the fact that Riemann's revolutionary insight flourished instead in a *non-geometrical* tradition dominated by the works of Christoffel and Ricci-Curbastro roughly in the same years. I will argue that, in the attempt to interpret general relativity as the last link of the chain Riemann–Helmholtz–Poincaré–Einstein, logical empiricists were led to argue that Einstein's theory of gravitation mainly raised a problem of *mathematical under-determination*, i.e. the discovery that there are physical differences that cannot be expressed in the relevant mathematical structure of the theory. However, a historical reconstruction of the alternative Riemann–Christoffel–Ricci–Einstein line of evolution shows on the contrary that the main philosophical issue raised by Einstein's theory was instead that of *mathematical over-determination*, i.e. the recognition of the presence of redundant mathematical differences that do not have any correspondence in physical reality.

1 Introduction

In the logical empiricists' philosophy of space and time, Einstein's conception of the relations between "geometry and experience" appears to be the heir of a 19th century philosophical and scientific tradition, the main protagonists of which were Riemann, Helmholtz, and Poincaré. The result of such a tradition appears to have

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come out most clearly in Reichenbach's celebrated theory of "equivalent descriptions". Riemann, Helmholtz, and Poincaré discovered the "principle of the relativity of geometry": we are free to choose among different metric geometries inasmuch as they can be transformed into one another by unique and continuous transformations, that is, insofar as as they are, in the Logical Empiricists' parlance, "topologically equivalent".

It is probably Michael Friedman who has most convincingly shown that such a tradition simply never existed (Friedman 1995). Helmholtz's and Poincaré's philosophy of geometry presupposes homogenous spaces, which can be covered by congruent tiles without gaps and overlappings. In such geometries there is a unique set of congruence relations, on which all observer can agree, or, more technically, a group of self-mappings with the properties of rigid motions can be defined. Riemann, on the contrary, left open the possibility of highly non-uniform spaces, where no group of motions can be defined and thus no unique conventional agreement can be made as to which tiles are congruent.

Logical empiricists were of course aware of the elementary fact that there are no rigid bodies in spaces of variable curvature. However, by stripping Helmholtz's and Poincaré's philosophy of geometry from their group-theoretical implications (Friedman 1995), they believed it was possible to simply shift the attention from "finite rigid bodies" to "infinitesimal rigid rods". As Roberto Torretti has shown, however, this strategy is hardly compatible with conventionalism. In a Riemannian manifold, an infinitesimal rod is considered rigid as long as it has a Euclidean behavior; one does not set by *convention* which rods are rigid, but instead checks it under the *hypothesis* that the space is Euclidean in its smallest parts (Torretti 1983, 235ff.).

In the Riemannian geometry adopted in general relativity, once a unit of measure has been arbitrarily fixed, the length of an infinitesimal measuring rod turns out to possess *an absolute value*, so that two intervals at a finite distance can be immediately compared. Einstein, it is true, refers, rather sporadically, to Helmholtz and Poincaré in his writings on the philosophy of geometry (Friedman 2002). However, Einstein's reference should be understood instead in the context of the so-called "measuring rod objection" against Hermann Weyl's attempt at unifying electricity and gravitation by dropping the length comparison "at-distance", rather then as a defense of conventionalism (Fogel 2008, ch. 3 and 4; Giovanelli 2012).

Thus it has been abundantly shown that the logical empiricists' attempt to read Einstein's general theory as the heir of a *geometrical tradition*, that starting with Riemann was developed into the epistemological works of Helmholtz and Poincaré, was substantially flawed. In my opinion, however, an even simpler historical point has escaped recent historical literature. Significantly, the logical empiricists were unable to philosophically appreciate the fact that Riemann's work, during roughly the same years, was mainly developed in a *non-geometrical tradition* in the work of authors such as Elwin Bruno Christoffel and Gregorio Ricci-Curbastro, the father of the so-called "absolute differential calculus" (our tensor calculus). Of course, Einstein himself explicitly considered general relativity as the direct heir of this tradition, or, as he famously put it, "a real triumph of the method of the general differential calculus" (Einstein 1915d, 778).

Riemann's work, considered from the point of view of Helmholtz's and Poincaré's philosophical reflections on geometry, appeared to the Logical Empiricist as concerned with the question of choosing between *alternative physical geometries*. Interpreted in the light of Christoffel's and Ricci-Curbastro's work, Riemann's main concern appears to be that of discerning the objective geometrical properties of the *same physical geometry*—those that are independent of the particular coordinate system we choose—from those properties that are a mere artifact of the coordinate system used.

As far I can see, the names of Christoffel and Ricci are not even mentioned by the logical empiricists. I would like to venture that this may at least partially have to do with the fact that they never intervened in the philosophical debate on the foundation of geometry. In order to show that some quantity has geometrical substance and is not just an artifact of some arbitrary choice of coordinates, they invoked the abstract study of the law of transformation of the quadratic differential forms. However, it is only from this point of view that the mathematical apparatus of Riemannian geometry and most of all its role in general relativity—in the form of the requirement of "general covariance"—can be understood.

The aim of this paper is to show that the inadequacy (Friedman, 1983; Nerlich 1994; Ryckman 2007, 2008) of the Logical Empiricists' interpretation of general relativity is in many respects the consequence of their failure to recognize the philosophical significance of this mathematical tradition. Logical Empiricists tried to interpret the role of Riemannian geometry in Einstein's general relativity through the lenses of the epistemological problems raised by Helmholtz and Poincaré. Historically and systematically, however, Riemann's revolutionary approach became part of Einstein's theory of gravitation through the mediation of the analytical work of Christoffel and Ricci.

After a brief description of how Riemann's insight was developed analytically (mainly, even if, of course, not solely) by Christoffel and later by Ricci in the so-called "absolute differential calculus" (Sect. 2), and then implemented in the general theory of relativity (Sect. 3), this paper concentrates on Reichenbach's famous "conversion" to conventionalism (Sect. 4) and on the emergence of the standard logical empiricist interpretation of the relation between geometry and physics (Sect. 6).

Even if the main claim of the paper could be extended to Logical Empiricists in general, Reichenbach's position appears particularly significant and will be treated as a sort of case study. Reichenbach's insistence on the "relativity of coordinates" in his first "Kantian" monograph (Reichenbach 1920b) appears to be much more effective in hindsight than his later appeal to the "relativity of geometry" (Reichenbach 1928). Reichenbach's deep knowledge of the mathematical apparatus of general relativity shows that this change of position was not the consequence of a trivial "blunder", but the conscious pursuit of a philosophical program. The philosophical inadequacy of his reading emerges paradoxically by following the more expository/semi-technical parts his work, rather than concentrating on his philosophical interpretation.

Recent historically-oriented philosophy of science has insisted on the importance of the 19th century debate about the foundation of geometry in order to understand the emergence of Logical Empiricism and, in particular, of its interpretation of the Theory of Relativity (Ryckman 1992; Coffa 1991; Friedman 1999; Howard 1994; Friedman 2008, only to mention some titles). However, in my opinion, even well-informed and influential works (see of instance DiSalle 2006) have not given sufficient attention to the implicit philosophical significance of the development of Riemann's ideas in the work of Christoffel and Ricci. This investigation was therefore left exclusively to the history of mathematics (see for instance the classical Reich 1994), rather than being considered a part of the "synthetic" history (Dickson and Domski 2010) that eventually led to the emergence of modern philosophy of science.

The filling of this lacuna in the historical literature on Logical Empiricism could help to clarify a more general philosophical misunderstanding, which seems to characterize early philosophical interpretations of general relativity (cf. Ryckman 2008, for an overview). General relativity, considered as the heir to the 19th century conventionalism of Helmholtz and Poincaré, seemed to raise a problem of *mathematical under-determination*; it shows the existence of mathematically equivalent, but *different physical geometries*, among which we can only make a pragmatic choice. Considered as the heir to the work of Riemann, Christoffel, and Ricci, general relativity appears to mainly raise a problem of *mathematical over-determination*; it shows that it is possible to represent the *same physical geometry* in different coordinate systems, using different mathematical functions. General Relativity shows the existence of a "redundancy" in the mathematical description, rather than "reducing" the mathematical structure that is physically relevant (Sect. 7)

2 Riemann's "Äquivalenzproblem" and its Analytical Development in the Work of Christoffel and Ricci-Curbastro

2.1 Riemann: From the Habilitationsvortrag to the Commentatio Mathematica

On June 10, 1854, before the Philosophical Faculty at Göttingen, Riemann delivered his celebrated *Habilitationsvortrag*, *Ueber die Hypothesen, welche der Geometrie zu Grunde liegen*. This short address would turn out to be one of the most influential (even if only partially understood) papers in the second half of the nineteenth century.

Riemann famously considered space as a special case of "n-dimensioned manifoldness" (Riemann 1868, 138; tr. Riemann 1873; cf. Scholz 1982), expressed by means of "n variables $x_1, x_2, x_3, \ldots, x_n$" (Riemann 1868, 139; tr. 1873, 15). Inspired by Gauss's theory of curved surfaces (Gauss 1827), Riemann assumed as a *hypothesis* (as the simplest among other possible alternatives) that space is distinguished from other "manifoldnesses" by the fact that the so-called line element ds, "is the square root of an always positive integrable homogeneous function of the second order of the quantities dx, in which the coefficients are continuous functions of" the quantities x" (Riemann 1868, 140; tr. 1873, 15; cf. Libois 1957; Scholz 1992). As is well known, Riemann's abstract approach turned out to be extremely powerful, allowing an infinity of possible geometries. Different geometries correspond to different expressions for the line element represented by different sets of coefficients.

The opposite, however, is not necessarily true. Riemann observed that every such expression can be transformed "into another similar one if we substitute for the n independent variables functions of n new independent variables" (Riemann 1868, 140; tr. 1873, 16), $x_1, \ldots, x_n$. The coefficients of the quadratic expression will depend on the variable used, but it is possible to show how the coefficients transform under a change of the independent variables in such a way as to make ds^2 remain unchanged.

However, as Riemann immediately made clear, "we cannot transform any expression into any other" (Riemann 1868, 140; tr. 1873, 16). The fact that a sphere cannot be projected onto a plane without distortion can be expressed analytically by the fact that it is impossible to convert the quadratic differential form, which holds on a sphere by means of a mere transformation of the independent variables, to one "in which the square of the line-element is expressed as the sum of the squares of complete differentials", that is, to one "in which the line-element may be reduced to the form $\sqrt{\sum dx^2}$" (Riemann 1868, 141, tr. 1873, 16; for more details Portnoy 1982; Zund 1983).

Different geometries are expressed analytically by different line elements, but the difference in the appearance of the line element does not necessary imply a geometrical difference. One of the main problems raised by Riemann's inquiry was discerning the geometrical properties that do not depend on a particular choice of the independent variables from those that are a mere deceptive appearance introduced by the mathematical formalism.

Riemann's lecture, which was intended for an audience of non-mathematicians, was intentionally scarce in the use of mathematical formulas. However, Riemann's somehow cryptic parlance is more familiar to the modern reader if one considers the notation that he introduced in the so-called *Commentatio mathematica* (Riemann 1876, tr. in Farwell 1990). The paper was submitted to the Paris Academy in 1861 to compete for a prize relating to the conduction of heat in homogeneous bodies with constant conductivity coefficients. The prize was not assigned and the *Commentatio* remained unknown until 1876, when Richard Dedekind—who found it in Riemann's *Nachlaß*—published it in the first edition of Riemann's work.

In order to address the question posed by the academy, Riemann had to reduce a system of partial differential equations to its simplest form. The problem turned out to be equivalent to that of reducing a quadratic differential from $\sum a'_{i,i} dx_i dx'_i$ (where $a'_{i,i}$ represent conductivity coefficients), to the form $\sum dx_i^2$ with constant coefficients, by a mere change of the independent variables x_i (where $i = 1, 2, 3$) (Riemann 1876, 392, tr. Farwell 1990, 241). [1]

Since it would be tedious to try various transformations of variables to establish the possibility of such a reduction, Riemann wanted to find a general criterion of transformability (cf. Farwell, 1990). For this purpose he introduced a four-index symbol $(\imath\imath', \imath''\imath''')$ containing the first and the second partial derivatives of the functions $a_{i,i'}$ with respect to the x_i. Then he showed that a quadratic differential

[1] For sake of historical accuracy along the paper we will try to remain faithful to the original notations. used by the various authors considered.

form can be transformed into one with constant coefficients if the four-index symbol vanishes: $(\iota\iota', \iota''\iota''') = 0$ (Riemann 1876, 402, tr. Farwell 1990, 242). Riemann found a criterion for distinguishing between the case in which the non-constancy of the conductivity coefficients $a_{i,i'}$ is a mere appearance of the mathematical description and the case in which it corresponds to a real physical difference, i.e. to a thermally non-homogeneous body.

Riemann hints, although vaguely, at a geometrical interpretation of this mathematical apparatus. The quadratic form $\sqrt{\sum'_{\iota,\iota} b'_{\iota,\iota} ds_\iota ds'_\iota}$ can be regarded as the "line element in a more general space of n dimensions extending beyond the bounds of our intuition" (Riemann 1876, 435, tr. Farwell, 1990, 252). A "flat space" can be represented by a quadratic differential form, whose coefficients are non-constant, such as polar coordinates or more complicated curvilinear coordinates. This difference however does not have any geometrical meaning; in this case it is always possible to find a transformation of variables with which the form $\sum b_{\iota,\iota'} ds_\iota ds_{\iota'}$ can be transformed into one with constant coefficients $\sum_\iota ds_\iota^2$ (Riemann 1876, 435, tr. Farwell 1990, 252). In a non-flat space, on the contrary, such a transformation cannot be found.

The four-index symbol furnishes a precise mathematical criterion: if and only if it vanishes the non-constancy of the coefficients is merely an artifact of the system of variables used. If not, the non-constancy has, so to speak, geometrical substance. Geometrically the four-index symbol corresponds to the "curvature … at the point $(s_1, s_2, \ldots, s_n)$" (Riemann 1876, 435, tr. Farwell 1990, 252). In his 1854 lecture Riemann had famously shown that this is not necessarily constant, opening the possibility of spaces with variable curvature. Spaces of constant curvature are merely a special case, where "independence of bodies from position" is assured (Riemann 1868, 149; tr. 1873, 36).

The *Commentatio* seems to make clear, however, that Riemann's main concern was capable of being expressed in a purely *analytical* way—i.e. independently from its possible physical or geometrical interpretation—as the problem of the *equivalence of differential quadratic forms*. This problem, which is best known as the *Äquivalenzproblem*, exerted a profound influence on the later development of mathematics and physics, as the development of absolute differential calculus (Tonolo 1961; Struik 1993) as its implementation in Einstein's theory of relativity shows. In my opinion, however, this was not fully appreciated in the philosophical debate raised by the appearance of Einstein's theory of gravitation. The work of Riemann, on the contrary, was read uncritically under the light of Helmholtz's and Poincaré's philosophy of geometry, who, as we shall see, were concerned with quite different philosophical questions.

2.2 A Very Brief History of the Emergence of the Absolute Differential Calculus: Christoffel, Ricci and Levi-Civita

Riemann's *Habilitationsvortrag* was discovered in the late 1860s by Richard Dedekind, who had been entrusted with Riemann's *Nachlaß*, and it was published in the "Abhandlungen der Königlichen Gesellschaft der Wissenschaften zu Göttingen," vol. 13, 1868 (Riemann 1868). Immediately afterward, Helmholtz, who had known of the existence of Riemann's paper in 1868 from Ernst Schering (Koenigsberger 1906, II, 139), published his famous *Über die Tatsachen, die der Geometrie zu Grunde liegen*, which appeared a little later in the "Göttinger gelehrte Nachrichten," vol. 15, 1868 (Helmholtz 1868).

Helmholtz famously derived Riemann's *hypothesis* that metric relations are given by a quadratic differential form (and not, for instance, by a quartic differential form) from the *fact* that there are rigid bodies, whose translations and rotations—expressed analytically by a set of differential equations—necessarily leaves a quadratic differential form unchanged. As is well known, Helmholtz's approach based on the notion of rigid body was enormously successful in the history of the philosophy of geometry.

As early as 1870, Helmholtz himself discussed the epistemological implications in the less technical and widely read talk *Über den Ursprung und die Bedeutung der geometrischen Axiome* (Helmholtz 1883) and in other philosophical papers that followed (Helmholtz 1878, 1879). In 1886, Sophus Lie (Lie 1886), stimulated by Felix Klein, reinterpreted and corrected Helmholtz's result on the basis of his theory of continuous groups. In 1887, Henri Poincaré (who already insisted on the group-theoretical approach to the concept of rigid motion in the early 1880s; Poincaré 1882 § 2), referred to Lie's results in his first paper on the foundations of geometry (Poincaré 1887, 214). At the end of the paper, Poincaré hints at the "celebrated *Memoire* of Riemann", in which every geometry is characterized "through the expression of the arc element as a function of the coordinates" (Poincaré 1887, 214). However, he discarded it as geometrically irrelevant, because it allows for spaces which exclude "the existence of a group of motion which does not alter distances" (Poincaré 1887, 214, see also Poincaré 1891, 773).

In general, Riemann's speculations about variably curved spaces (with the notorious exception of Clifford 1876) were either ignored or dismissed (Hawkins 1980, 2000). Instead, Riemann's paper triggered developments in the non-geometrical branch of mathematics concerned with the study of quadratic differential forms. Dedekind had mentioned Riemann's unpublished *Habilitationsvortrag* to Erwin Bruno Christoffel, who in 1862 filled Dedekind's post at ETH in Zurich (Butzer 1981).

In the last paragraph of the paper published in 1869 in the "Journal für die reine und angewandte Mathematik" (the celebrated Crelle's Journal), *Über die Transformation der homogenen Differentialausdrücke zweiten Grades* (Christoffel 1869), Christoffel in fact referred briefly to "an essay [Abhandlung] in Riemann's *Nachlass* to which Mr. Dedekind has announced to provide the missing analytical elaborations" (Christoffel 1869, 70), which, of course, had already been published (Riemann 1868)

Christoffel's paper addressed the equivalence problem for two quadratic differential forms in the most general way, without focusing on the special case of the reducibility to an expression with constant coefficients (Ehlers 1981).

Christoffel wanted to establish which "conditions are necessary [erforderlich]" (Christoffel 1869, 46), for transforming a quadratic differential form $F = \sum \omega_{ik} \vartheta x_i \vartheta x_k$

into another such form $F' = \sum \omega'_{ik}\vartheta x'_i \vartheta x'_k$ by means of a smooth, invertible substitution of the independent variables. In order to answer this question, Christoffel was led "for a better overview [zur besseren Übersicht]" (Christoffel 1869, 48) to express the recurrent combination of the ω_{ik} and their first partial derivatives $\frac{\partial \omega_{gk}}{\partial x_h}$ through two kinds of three-index symbols $\begin{bmatrix} gh \\ k \end{bmatrix}$ and $\left\{ \begin{matrix} il \\ r \end{matrix} \right\} = \sum \frac{E_{rk}}{E} \begin{bmatrix} gh \\ k \end{bmatrix}$[2] (the now famous Christoffel symbols respectively of first and second kind; Reich 1994; Herbert 1991).

The purely "algebraic" criterion for the equivalence of two differential forms is then obtained by introducing a four-index symbol $(ghki)$ that can be constructed from the "Christoffel symbols" and their derivatives (that is, from the first and second partial derivatives of the ω_{ik}, Christoffel 1869, 54). It corresponds to Riemann's four-index symbol in the *Commentatio* (which Christoffel could not have known). Two quadratic forms can be transformed into one another only if (locally) $(ghki) = (ghki)'$.

With the exception of a hint at the problem of the developable surfaces (Christoffel 1869, 47), no reference to the geometrical concept of "curvature" can be found in Christoffel's paper, which follows a purely algorithmic approach. This attitude was taken up by Gregorio Ricci-Curbastro, who in six papers published between 1883–1888 was able to develop systematically Christoffel's solution of the *Äquivalenzproblem* into a new calculus (Dell'Aglio, 1996; Bottazzini 1999). In his first paper on the argument, *Principii di una teoria delle forme differenziali quadratiche* (Ricci-Curbastro 1883), Ricci recognized his debts to Riemann (Ricci could now refer to Riemann's *Commentatio*), Rudolf Lipschitz (Lipschitz 1869) and most of all Christoffel.

Like Christoffel, Ricci was not interested in making a contribution to geometry. According to Ricci, "mathematicians have usually considered quadratic differential forms … as representing line elements of n-dimensional spaces" (Ricci-Curbastro 1883, 140). This, however, has often led to confusion. Ricci explicitly emphasized that his own investigations were based "on purely analytical concepts", leaving aside "the rather vacuous [oziose] discussions about the existence of spaces more than three dimensions" (Ricci-Curbastro 1883, 140).

Ricci's goal was to develop a purely abstract theory of differential invariants, a calculus of quadratic differential forms, such as "$\varphi = \sum_{rs} a_{rs} dx_r dx_s$ where a_{rs} are functions of $x_1, x_2, \ldots, x_n$." (Ricci-Curbastro 1886, 3). The problem was then to establish the laws according to which the coefficients would transform by a change in the independent variables when "one substitutes the variables $x_1, x_2, \ldots, x_n$ with the variables $u_1, u_2, \ldots, u_n$" (Ricci-Curbastro 1886, 4) (which are smooth functions of the first ones) under the condition that $\varphi = \varphi'$.

Ricci showed that the coefficients a_{rs} transform according to certain rules into the new coefficients (a_{pq}), so that the new form is called "covariant" with respect to the first; the reciprocal form $a^{(rs)}$ transformed "contravariantly" into $a^{(pq)}$ (Ricci-

[2] where $\frac{E_{rk}}{E}$ are the inverse matrix of ω_{ik}

Curbastro 1889, 113). Hence in Ricci's parlance a_{rs} and $a^{(rs)}$ form respectively a covariant and contravariant "system" of second rank (because of the two indices).

To establish the conditions of tranformability, Ricci followed Christoffel's path of introducing the three index-symbol $a_{rs,i}$ (for the Christoffel symbols of first kind), and the four index-symbol $a_{hi,jk}$ which he later called "the system of Riemann" (Levi-Civita and Ricci-Curbastro 1900, 142). However, Ricci, beginning with his 1887 paper *Delle Derivazioni covarianti e controvarianti e del loro uso nella analisi applicata* (Ricci-Curbastro 1888) interpreted Christoffel's algorithms as a differentiation of a more general kind that he labeled as "covariant (contravariant) differentiation". (Ricci-Curbastro, 1888, 4). With the repeated application of the covariant differentiation, starting from a covariant (or contravariant) primitive system, others can be obtained (Ricci-Curbastro 1888, § 2).

This was the necessary step that allowed Ricci to transform Christoffel's still unsystematic approach into a new "calculus". In 1893, Ricci called it "absolute differential calculus" for the first time, where the word "absolute" expresses the fact that the calculus can be applied "independently of the choice of the independent variables" and requires "that the latter are completely general and arbitrary" (Ricci-Curbastro 1893, 311). Ricci made his results known outside of Italy with a summary published in French in Georges Darboux's "Bulletin des Sciences Mathématiques" in 1892 (Ricci-Curbastro 1892). Only in 1901, with the assistance of his student Tullio Levi-Civita, did he publish a memoir in French, which can be considered the manifesto of Ricci's calculus, "Méthodes de calcul différentiel absolu et leurs applications" (finished in 1899) in Felix Klein's journal, "Mathematische Annalen" in 1900 (Levi-Civita and Ricci-Curbastro, 1900).

3 "A Real Triumph of the Method of the General Differential Calculus": Einstein's General Relativity

Ricci's calculus apparently failed to find his audience among differential geometers (Roth 1942, 266; Reich 1994, 77, but see Bottazzini 1999). Ricci's algorithms appeared incapable of providing any substantially new geometrical results that could not have been reached through a more traditional approach. This was what Luigi Bianchi, author of a celebrated handbook on differential geometry (Bianchi 1894), meant when reviewing Ricci's work for the royal prize of the Accademia dei Lincei, by characterizing it as "useful but not indispensable" (Bianchi 1902, 149; on the relations Bianchi-Ricci see Toscano 2001).

It is usually argued that only general relativity eventually did justice to Ricci's work. Einstein's progressive appropriation of the work of Riemann, Christoffel, Ricci, and Levi-Civita has of course been discussed several times in historical literature (see for instance: Earman and Glymour 1978b; Stachel 2002; 1984; Janssen and Renn 2007; Pais 1982, 212ff.). Here we will only give a superficial presentation in order to emphasize some elements that will be relevant to understanding the subsequent philosophical discussion. The secondary literature listed below is of course far from being exhaustive and does not reflect priorities.

3.1 Einstein, Grossmann and the Absolute Differential Calculus

In 1907, Einstein put forward the principle of equivalence as an extension of the relativity principle to uniformly accelerated systems (Einstein 1907). After having embraced Minkowski's geometrical interpretation of special relativity, in 1912 Einstein, as he later recalled (Einstein 1922, EA 1-14), had grasped the "decisive idea": the analogy between his work on extending the principle of relativity to accelerated motion with Gauss's theory of surfaces, "without being aware at that time of the work of Riemann, Ricci, and Levi-Civita" (Einstein 1922, EA 1-14).

Famously, in August 1912, Marcel Grossmann introduced Einstein to this mathematical tradition: "never before in my life," as Einstein wrote in a letter to Arnold Sommerfeld in the October of the same year, "have I gained enormous respect for mathematics, whose more subtle parts I have considered until now, in my ignorance, as pure luxury" (CPAE, Vol. 5, Doc. 421)

Einstein and Grossmann jointly published the *Entwurf einer verallgemeinerten Relativitätsheorie und einer Theorie der Gravitation* (Einstein and Grossmann 1913). Einstein's "Entwurf" theory (just as his final general theory of relativity) is built around a quadratic differential form $ds^2 = \sum g_{\mu\nu} dx_\mu dx_\nu$ that assumes the name of "metric tensor" or "fundamental tensor" (which corresponds to Ricci's "covariant system of second rank" Einstein and Grossmann 1913, 25, n.). The physical novelty consisted of course in the fact that the coefficients $g_{\mu\nu}$ of the quadratic differential form represent the behavior of measuring rods and clocks with reference to the coordinate system, as well as the potentials of the gravitational field. The geodesic trajectories of particles can be considered as the solutions to the variational problem $\delta\{\int ds\} = 0$.

Grossmann/Einstein's problem was then to find the "differential equations" to determine "the quantities $g_{\mu\nu}$ i.e., the gravitational field" (Einstein and Grossmann 1913, 11). Einstein's strategy was famously to look for a "generally covariant" analogon of the Poisson's equation $\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial y^2} + \frac{\partial^2\varphi}{\partial z^2} = 4\pi k\rho$. The ten potentials $g_{\mu\nu} = g_{\nu\mu}$ play the role of the single potential φ, whereas the density ρ corresponds to the ten components of a second rank tensor $\Theta_{\mu\nu}$, the so-called stress-energy tensor. A second rank tensor $\Gamma_{\mu\nu}$, constructed only from the $g_{\mu\ \nu}$ and their first and second derivatives with respect to the coordinates (just like Poisson's equation involves the second derivative of the potential), should play the role of the gravitational tensor: Thus the field equations "would likely have the form $\kappa \cdot \Theta_{\mu\nu} = \Gamma_{\mu\nu}$ where κ is a constant" (Einstein and Grossmann 1913, 11).

As the Zurich Notebook reveals (page 14L), Grossman had immediately found a "tensor of fourth manifold" (Tensor vierter Mannigfalfigkeit)—(*ik*, *lm*) using the four-index-symbol notation—as the only tensor that contains only the metric tensor and its first and second derivatives (CPAE 4, Doc. 10; for an extensive commentary of the notebook see Renn 2007, vol. 1. and 2.). It turned out to be much more complicated to find the exact form of the "covariant differential tensor of second rank and second order" (Einstein and Grossmann 1913, 36), obtained from the Riemann–Christoffel tensor by contraction (that is, by setting unlike indices equal) that could play the role of the gravitational tensor (having the same valence and rank of the matter tensor).

Einstein and Grossman discarded the natural candidate, the so-called Ricci–Tensor (see for instance: Maltese 1991), since "it does *not* reduce to $\Delta\phi$", that is to the Newtonian limit (assumed erroneously as spatially flat), "in the case of weak static field" (Einstein and Grossmann 1913, 337). In subsequent years Einstein gave up general covariance for the equations of gravitational field. An argument, which came to be known as the "hole argument", even convinced Einstein that, as he wrote to Paul Ehrenfest in early 1914, "generally covariant field equations that determine the field completely from the matter tensor cannot exist at all" (CPAE, Doc. 512, 5, 563; see for instance Norton 1987). In March of the same year Einstein even wrote to Michele Besso that "[t]he general theory of invariants only acted as a hindrance" (CPAE, 5, Doc. 514, 604).

In October 1914, Einstein, who in the meantime had moved to Berlin, presented a systematic exposition of the *Entwurf* theory to the Berlin Academy entitled, *Die formale Grundlage der allgemeinen Relativitätstheorie* (Einstein 1914). Section 7, "Geodesic line or equations of the point motion", introduced a fundamental formal innovation, using the absolute differential calculus to express the equation of the geodesic line independently of the coordinate using the Christoffel symbols (Einstein 1914, 1044ff.):

$$\frac{d^2 x_\tau}{ds^2} = \sum_{\mu\nu} \left\{ \begin{matrix} \mu\nu \\ \tau \end{matrix} \right\} \frac{dx_\mu}{ds} \frac{dx_\nu}{ds}$$

After giving a presentation of the "hole argument" in § 12, in § 13 Einstein still restricts the covariance of the field equations in order to ensure a unique relation between $g_{\mu\nu}$ and $T_{\mu\nu}$ (Einstein 1914, 1066ff.).

3.2 Generally Covariant Field Equations

In the meantime the theory, which had initially been received rather suspiciously by physicists, began to attract the interest of mathematicians such as Levi-Civita himself (Cattani and De Maria, 1989) and David Hilbert (Mehra 1974; Earman and Glymour 1978a; Sauer 1999; Corry 2003). Under the pressure of a hasty competition with the latter, by November of 1915 Einstein regained general covariance for his field equations, which he had abandoned "with a heavy heart" (Einstein 1915d, 778) three years before, and presented them in four communications to the Prussian Academy (Einstein 1915a, b, c, d). In the first of three papers presented to the Berlin Academy he famously described the general theory of relativity as: *"a real triumph of the method of the general differential calculus founded by Gauss, Riemann, Christoffel, Ricci, and Levi-Civiter* [sic]" (Einstein 1915d, 778; my emphasis).

Einstein proceeded very roughly as follows. The righthand side of the field equations, the matter tensor $T_{\mu\nu}$, is a second rank and—because of the conservation of energy-momentum—divergence-free tensor. The "Ricci tensor" $R_{\mu\nu}$—which Einstein considered again as the reasonable candidate for the lefthand side of his equations—is a two-index-object as well, but its divergence is generally non-zero. Einstein discovered that the tensor $G_{\mu\nu}$ obtained by subtracting the term $\frac{1}{2}Rg_{\mu\nu}$ (where R is the trace of Ricci tensor $g^{\mu\nu}R_{\mu\nu}$) from the Ricci tensor $R_{\mu\nu}$ is divergence-free. The tensor $G_{\mu\nu}$ (later labeled "Einstein tensor") is therefore suitable to take on the role of the gravitation tensor of the final field equations: $G_{\mu\nu} = -\kappa T_{\mu\nu}$ (Mehra 1974, 20; Pais 1982, 20). As Einstein explained to Hilbert in November 1915:

> The difficulty was not in finding the generally covariant equations from the $g_{\mu\nu}$ for this is easily achieved with the aid of the Riemann's tensor. Rather, it was hard to recognize that these equations are a generalization, and precisely, a simple and natural generalization of Newton's law. It has just been in the last few few weeks that I succeeded in this (I sent you my first communication) [3] whereas 3 years ago with my friend Grossmann, I had already taken into consideration the only possible generally covariant equations, which have now been shown to be the correct ones (Einstein to Hilbert, November 18, 1915; CPAE 8a, Doc. 148, 201).

An important step toward the final breakthrough is usually considered the overcoming of what Einstein famously called a "fateful prejudice" [ein verhängnisvolles Vorurteil] (Einstein 1915d, 782) "The key to this solution" was found, as he wrote in a letter to Sommerfeld, when Einstein (starting in November 1915) came to realize that the negative Christoffel-symbols (of the second kind) $\Gamma^{\tau}_{\mu\nu} = -\left\{ \begin{matrix} \mu\nu \\ \tau \end{matrix} \right\}$ are "to be regarded as the natural expression of the gravitational field 'components"' (28.11.1915; CPAE 8a, Doc. 153, 207–208; Norton 2003; Janssen 2008).

A free material point moves with uniform motion in a straight line, relative to a system in which the $g_{\mu\nu}$ are constant $(\Gamma^{\tau}_{\mu\nu} = 0)$. If a new system of coordinates is smoothly introduced, the $g_{\mu\nu}$ will no longer be constant, but will be functions of the coordinates $(\Gamma^{\tau}_{\mu\nu} \neq 0)$; the motion of the free material point will present itself in the new co-ordinates as curvilinear non-uniform motion. Via the principle of equivalence, we can equally well interpret this motion as a motion under the influence of a gravitational field.

As Einstein summarizes in the final 1916 "polished" presentation of the theory, published in the *Annalen der Physik* (Einstein, 1916b), it is then natural to extend this reasoning to the case "when we are no longer able by a suitable choice of co-ordinates to apply the special theory of relativity to a finite region" (Einstein 1916b, 779), that is, when the $\Gamma^{\tau}_{\mu\nu}$ cannot be made to vanish identically, since the Riemann–Christoffel tensor $R^{\rho}_{\mu\nu\tau}$ does not vanish (Norton 1985).

Interestingly, even if Einstein does refer to the non-Euclidean geometry, in his 1916 paper the Riemann–Christoffel tensor $R^{\rho}_{\mu\nu\tau}$ does not carry most of the geometrical implications that we take for granted today; in particular Einstein does not refer to the "curvature of spacetime" (Reich 1994, 204f.). The Riemann tensor

[3] Einstein 1915d

is introduced insofar as it is the only tensor that can be constructed solely from the fundamental tensor without going beyond the second derivatives of the $g_{\mu\nu}$ (as the analogy with the Poisson equations requires).

A "geometrical" issue emerges instead when one has to compare the values of the $g_{\mu\nu}$ predicted by the field equations (let us say the Schwarzschild-solution, Schwarzschild 1916) with the observed values. Until the end of his life Einstein insisted on assuming "provisionally" (Howard 1990, 2005) that these can be obtained by direct measurement using small rods and atomic clocks, the length and rate of which are independent of the gravitational field they are embedded in (Stachel 1989). Roughly, the $g_{\mu\nu}$ are the numbers to which we have to multiply the coordinate distances so that ds^2 has the same absolute value (up to the global choice of unit of measure) everywhere on the manifold.

3.3 Critique and Geometrical Development of Absolute Differential Calculus

As early as 1916, the Austrian physicist Friedrich Kottler criticized Einstein for referring to the Christoffel symbols as the components of the gravitational field (Kottler 1916). Christoffel Symbols are not tensors, and they can be non-zero in a flat spacetime simply by virtue of curvilinear coordinates. Responding to Kottler, Einstein suggested that the equation $\frac{d^2x_\tau}{ds^2} = \Gamma^\tau_{\mu\nu}\frac{dx_\mu}{ds}\frac{dx_\nu}{ds}$ "as a whole is generally covariant" (Einstein 1916a, 641), but the two terms taken separately are not. In particular the first term of the geodesic equation can be taken as representing "the Galilean inertia", and the second term with the Christoffel symbols, as "representing influence of the gravitational field upon the mass point" (Einstein 1916a, 641). Neither of these *per se* has physical meaning, only their "sum" does.

In Einstein's view the covariance principle and the equivalence principle appear then to be deeply connected: the variability of $g_{\mu\nu}$ and the non-vanishing of the $\Gamma^\tau_{\mu\nu}$ introduced by a coordinate transformation can be interpreted arbitrarily as an acceleration field or as a (homogeneous) gravitational field: "the requirement of general covariance of equations embraces the principle of equivalence as a quite special case" (Einstein 1916a, 641). Einstein explicitly embraced the view that the gravitational field is a coordinate-dependent quantity (Einstein 1918a, 699f.; see Janssen 2011).

The most famous objection against Einstein's use of the absolute differential calculus was raised of course by Erich Kretschmann in 1917 (Kretschmann 1917). The principle of "general covariance" as complete coordinate generality in the formulation of a physical theory has no particular physical content, and thus it has nothing to do with a principle of relativity. In fact "according to Ricci and Levi-Civita's investigations" (Kretschmann 1917, 579) every space-time theory can be formulated in a generally co-variant way, only by inserting the $g_{\mu\nu}$ and the $\Gamma^\tau_{\mu\nu}$ into the equations of the theory (for more on this topic see Rynasiewicz 1999; Norton 2003). In a 1918 paper, Einstein agreed with Kretschman's claim, emphasizing nevertheless the heuristic value of general covariance when combined with a principle of simplicity (Einstein 1918c).

A more compelling answer was provided by Hermann Weyl in the first edition of *Raum–Zeit–Materie* (Weyl 1918b). Even if every theory can be rewritten in a generally covariant form, in pre-general-relativistic theories such as special relativity, the metric displays the pre-assigned Minkowski values g_{ik} and the Γ^i_{hl} vanish everywhere; in general relativity one finds these values only after having solved the field equations: "This seals the doom of the Idea that a geometry may exist independently of physics in the traditional sense" (Weyl 1918b, 174).

Weyl could make this point clear by exploiting the geometrical implications of tensor calculus that were developed in those years under the stimulus of general relativity. In 1916/1917, Levi-Civita (Levi-Civita 1916, but see also Hessenberg 1718 and Schouten, 1919) had famously recognized the geometrical meaning of the Christoffel symbols as determining the parallel displacement of vectors (Struik 1989; Reich 1992). A displacement preserving the direction of a vector can be expressed in general coordinates precisely by the fact that the Christoffel symbols can be made to vanish along the path; thus roughly, referring to the displacement operation, the Γ^i_{hl} turned out to be expressible without referring to g_{ik}.

The absolute differential calculus can then be the founded "geometrically". When a vector in a Euclidean space is parallel-transported around a loop, it will always return to its original position (Γ^i_{hl} vanish overall on the manifold). However, this property does not hold in the general case. The Riemann curvature tensor measures precisely the change in the direction (but not in magnitude as in Weyl's more general non-Riemannian geometry Weyl 1918c) of a vector after it is transported around a closed loop (Weyl 1918b, § 16). Along a geodesic path the vector will remain 'unchanged', so that a geodesic line can be defined in a non-metrical way as the straightest line rather than the line of extremal length (Weyl 1918b, § 17).

In the third revised edition of *Raum–Zeit–Materie* published in 1919, the Christoffel Symbols $\Gamma^i_{sr} = \Gamma^i_{rs}$ are considered as the "components of an *affine connection*" (Weyl 1919, 101). Precisely because it is not a tensor, the affine connection provides an adequate representation of the fact required by Einstein's interpretation of the equivalence principle: there is no unique decomposition of the connection into an inertial background plus a gravitational field (Stachel 2007).

In 1920, Weyl introduced the celebrated expression "guiding field" (*Führungsfeld*) for the affine connection: general relativity does make every motion relative, eliminating the structure responsible for the distinction between geodesic and non-geodesic worldlines; Einstein's theory instead transformed such a constraining "guidance" in a physical force-field, in which "according to Einstein, inertia and gravitation constitute an inseparable unity" (Weyl 1920). Motion along a geodesic path or deflection from a geodesic path are absolutely different, but can be interpreted arbitrarily as the effect of inertia or gravitation.

In 1920, in the unpublished paper *Grundgedanken und Methoden der Relativitätstheorie in ihrer Entwicklung dargestellt* (CPAE 7, Doc. 31), Einstein, using a celebrated analogy with the electromagnetic field, which will be split differently into electric and magnetic components by different observers, came to recognize that the crucial point of general relativity is the fact that "the gravitational field only has a *relative existence*" (CPAE 7, Doc. 31, p. 21; Janssen 2005, 2008). In the Princeton Lectures, published in the same year, Einstein emphasized again that this is well represented by the fact that "the intensity of the gravitational field" is expressed by the quantities $\Gamma^{\mu}_{\alpha\beta}$,which "do not have a tensor character" (Einstein 1921b, 52) and thus are coordinate-dependent.

In the early 1920s, Élie Cartan, starting from Levi-Civita's geometrical notion of parallel displacement, considered gravity, mathematically represented by the affine connection, as the structure reconciling the different orientations of local inertial frames (Cartan 1923, 1924b, 1925). From this point of view, according to Cartan, "relativity faces the paradoxical task of interpreting, in a non-homogeneous universe, all the results of so many experiences by observers who believe in homogeneity of the universe" (Cartan 1924a, 81). At this point, it might be said, the connection of Ricci's calculus with Riemann's original geometrical approach is restored. With general relativity, as Levi-Civita noticed some years later, clearly hinting at Bianchi's words, "Ricci's calculus revealed itself to be not only useful but truly indispensable" (Levi-Civita 1925, 11, tr. 1927, VII).

4 Relativity of Coordinates Versus Relativity of Geometry: The Young Reichenbach's Conversion to Conventionalism

On the philosophical side, in as early as March 1917 Schlick had published an article version of his classical *Raum und Zeit in der gegenwärtigen Physik* (Schlick 1917a). The work appeared in the same year in book form (Schlick 1917b), which was published in four different editions up until 1922 (see Schlick 2006, vol I, 2). Schlick, as is well known, exploited the geometrical implications of General Relativity in a quite different way, by casting general relativity in a conventionalist tradition, which Schlick had allegedly found in the work of Riemann, Helmholtz and, most of all, Poincaré. Space-time is in itself metrically "amorphous [gestaltlos]", as Poincaré has argued (Schlick 1917a, 167); a certain space or space-time is indistinguishable from every other by a continuous and one-to-one transformation that preserves the neighborhood relations among points or events. Hence the choice among them can only be made by an arbitrary stipulation, an implication which Schlick called "the geometrical relativity of space" (Schlick 1917a, § II).

Einstein, as it is well known, was very pleased by Schlick's paper and his opinion was similarly very positive for the book version (Howard 1984). However, in as early as 1920, in his first monograph on relativity, Hans Reichenbach, who had attended Einstein's lectures on general relativity in Berlin in the late 1910s, had raised a rather convincing objection against a conventionalist approach to general relativity.

Conventionalism, Reichenbach argued, works only for spaces of constant curvature. In Riemannian geometry of variable curvature no unique set of congruence relations can be defined all over the space, so the very idea of a unique conventional choice among alternative congruent relations does not make sense. For this reason, Reichenbach points out, Poincaré "excludes from the beginning Riemannian geometry, because it does not permit the displacement of a body without change of form" (Reichenbach 1920b, 104, n. 1; tr. 1965, 109, n. 1; translation modified). In the general case, only the *unit of length* is globally available on a Riemannian manifold (per convention, all observers can agree to use, for instance, the centimeter as the unit of measure). In contrast, in in the non-Riemannian geometry of Weyl a separate unit of length at every point of space may be defined.

According to Reichenbach, Einstein's general theory, adopting the Riemannian approach, leads rather "to an *absolutely objective determination of the structure of space*" (Reichenbach 1920a, 405; tr. 2006, 29). Even if we are free to choose the coordinate system, only the properties that are independent of a particular coordinate system are physically meaningful: "Relativity does not mean the abandoning of a judgment, but the liberation of the objective sense of knowledge from its *distortion through our subjective nature*" (Reichenbach 1920a, 405; tr. 2006, 29, on the importance of this topic see Ryckman 2005, § 2.4.4).

In these few words Reichenbach seemed to catch what general relativity had inherited from the Riemannian tradition: Riemannian geometry is formulated in such a way that it works in arbitrary coordinates. Whereas Schlick tried to cast the contribution of Riemannian geometry to general relativity in the light of Helmholtz's and Poincaré's philosophy of geometry as the freedom of choosing among *different physical geometries*, Reichenbach insisted that Riemann had showed under which condition it was possible to express *the same physical space-time* by different mathematical functions depending on which coordinates were used.

4.1 Relativity of Coordinates: Reichenbach's Early Interpretation of Riemannian Geometry and of its Role in General Relativity

Reichenbach's early approach reproduces quite well Einstein's original reasoning. As Einstein wrote to Schlick, "in principle there can exist finite (matter-free) parts of the world" (Einstein to Schlick, March 21, 1917; CPAE 8a 305) that can be covered by a rectangular grid of unit rods and clocks. The four-dimensional line element is expressed as the sum of the squares of the coordinate differential $ds^2 = \sum_1^4 dx_\nu^2$. If one introduces new curvilinear coordinates by means of an arbitrary smooth substitution of the independent variables, the line element will not preserve its simple form but will change into a mixed quadratic expression: $ds^2 = \sum_1^4 g_{\mu\nu} dx_\nu dx_\nu$:

> The coefficients $g_{\mu\nu}$ occurring in [the mixed quadratic differential form] manifest themselves in the acceleration of the second coordinate system relative to the inertial system; since this acceleration directly characterizes the gravitational field of the second system, we may regard it as a measure of this gravitational field. We notice, therefore, that the transition from a gravity-free field to a gravitational field is connected with a transition to *non-Euclidean coordinates*, and that the metric of these coordinates is a measure of the gravitational field (Reichenbach, 1920b, 23; tr. 1965, 24).

Reichenbach, however, is careful to emphasize that "[s]uch a space is *only apparently non-Euclidean*; actually it does not differ structurally from Euclidean space" (Reichenbach 1920b, 23; tr. 1965, 25; my emphasis). The non-Euclidean appearance of the line element does not necessarily imply a non-Euclidean geometry. It is, on the contrary, the very same flat space-time that "can be expressed in terms of *non-Euclidean coordinates*" (Reichenbach 1920b, 23; tr. 1965, 25; my emphasis), where the $g_{\mu\nu}$ are not constant but functions of the coordinates.

Einstein identified the presence of the gravitational field with the non-constancy of $g_{\mu\nu}$. As Reichenbach observes, the "transition is made from the special theory to the general theory of relativity" can be considered as "*a far reaching extrapolation*" (Reichenbach 1920b, 24; tr. 1965, 26; my emphasis). Einstein "inferred from this that *every gravitational field*, not only that which is produced by transformation, manifests itself by a deviation from Euclidean geometry" (Reichenbach 1920b, 23; tr. 1965, 24). The presence of gravitation manifests itself in the non-constancy of the $g_{\mu\nu}$, also in the case where it is not possible "to choose the coordinates in such a way that the line element becomes Euclidean at all points simultaneously" (Reichenbach 1920b, 27; tr. 1965, 29).

This is the significance of the introduction of a quadratic form with variable coefficients: (cf. Ryckman 2005, 35ff.):

> The special position of the mixed quadratic form of the line element can also be characterized in the following way. The ten functions $g_{\mu\nu}$ determining the metric *are not absolutely fixed*, but depend on the choice of the coordinates. They are not independent of one another, however, and if four of them are given, the coordinates as well as the other six functions are determined. This *dependence* expresses the *absolute character of the curvature of space*. The metric functions $g_{\mu\nu}$ are not relative; that is, *their choice is not arbitrary* (Reichenbach, 1920b, 27; tr. 1965, 29; my emphasis).

If one covers a flat space with "any curved oblique coordinates, then the line element will become a mixed quadratic expression. Even the ordinary polar coordinates furnish an expression differing from the pure quadratic sum for the line element" (Reichenbach 1920b, 24; tr. 1965, 25). Thus, even the very simple "representation of Euclidean space by means of polar coordinates can be conceived as a projection upon a non-Euclidean space" (Reichenbach 1920b, 24; tr. 1965, 25). In polar coordinates the $g_{\mu\nu}$ are not constant. However "the Riemannian measure of curvature of this system will be zero at every point" (Reichenbach 1920b, 23; tr. 1965, 25). The lines of the coordinate grid are curved, but not the surface itself. Cartesian coordinates, where the $g_{\mu\nu}$ has constant values, can be reintroduced by a simple coordinate transformation. By contrast, on a non-flat non-Euclidean space "it is impossible to preserve its simple Euclidean form" (Reichenbach 1920b, 94; tr. 1965, 99). Cartesian coordinates simply do not exist:

> the four *space-time coordinates can be chosen arbitrarily*, but that the *ten metric functions $g_{\mu\nu}$ may not be assumed arbitrarily*; they have definite values for every choice of coordinates… If the metric were a purely subjective matter, then the Euclidean metric would have to be suitable for physics; as a consequence, all ten functions $g_{\mu\nu}$ could be selected arbitrarily. However, the theory of relativity teaches that the metric is subjective only insofar as it is

> dependent upon the arbitrariness of the choice of coordinates, and that independently of them it describes *an objective property of the physical world* (Reichenbach 1920b, 86–87; tr. 1965, 90–91)

As Reichenbach recognizes, the mathematical apparatus of Riemannian geometry is mainly concerned with the problem of establishing when different sets of $g_{\mu\nu}$ represent different geometries, and when they are merely the consequence of the coordinate system used. As we have seen, Riemann, Christoffel and Ricci had found in the so called Riemann–Christoffel tensor an *absolute* criterion for distinguishing the class of different $g_{\mu\nu}$-systems that differs only by a coordinate transformation from other classes.

Hence, the freedom in the choice of the coordinate system has nothing to do with the freedom in the choice of geometry:

> It is true that the metric contains a *subjective element*, and depending on the choice of the system of reference, the metric coefficients will vary; this *indeterminacy* [Unbestimmtheit] still holds in the gravitational field. But there exist *dependency relations* among the metric coefficients, and if four of them are arbitrarily given for the whole space, then the other six are determined by transformation formulas. … That something exists manifests itself in the dependency relations between the metric coefficients; since *we can discover these relations by means of measurements—and only by means of them—we can discover the real*. It is the essence of the general theory of relativity that the metric is much more than a mathematical measurement of bodies; it is the form by means of which the body is described *as an element in the material world*. (Reichenbach, 1920b, 96–97; tr. 1965, 102; my emphasis)

Reichenbach insists on the fact that such *indeterminacy*, i.e. the existence of non-physical degrees of freedom, does not reflect any lack of determinacy of the geometrical structure of the world. Reichenbach then came to the conclusion that the metric contains a *subjective aspect which depends on the choice of the coordinate system* and *objective one*, which *is expressed in the dependencies among the metric coefficients*.

In a way somehow similar to that of Arthur Eddington (Eddington, 1920), the still "Kantian" Reichenbach emphasizes the epistemological significance of the use of tensor calculus. The philosophical meaning of the "Riemannian analytic metric" is that it presents the mathematical technique to isolate those elements that have objective physical significances, from those that are merely artifacts of the coordinates: "invariance with respect to the transformations characterizes the *objective content* of reality, the structure of reason expresses itself in the *arbitrariness of admissible systems*" (Reichenbach 1920b, 86; tr. 1965, 90; my emphasis).

4.2 Relativity of Geometry: Reichenbach's Last Step to Conventionalism

After having the opportunity to take a first look at Reichenbach's book, Schlick immediately wrote to Einstein: "Reichenbach does not seem to me to be fair [nicht gerecht zu sein] toward Poincaré's conventionalism [Konventionslehre]" (Schlick an Einstein, 9.10.1920; CPAE 10, Doc. 171). Writing to Reichenbach some weeks later, Schlick tried to debunk Reichenbach's critique of conventionalism by arguing that Poincaré in later writings included geometries of variable curvature in his approach (Reichenbach to Schlick 26.11.1920; Schlick and Reichenbach 1922). Reichenbach responded by agreeing on the fact that in principle one could choose between keeping relativity and abandoning Euclidean geometry or vice-versa: "physics, however, makes the *first* decision… you, and Poincaré, would say for the sake of simplicity [um der Einfachheit halber]… But I have an instinctive disinclination [Abneigung] for this interpretation" (Reichenbach to Schlick 29.11.1920; Schlick and Reichenbach 1922).

Schlick's arguments must have been very persuasive. Reichenbach quickly overcame his "instinctive disinclination" for conventionalism (on this point see: Parrini 2005; Dieks 2010). Einstein's epistemological achievement becomes precisely that he has shown it would have been possible in principle for physics to make the *second* decision, that is, to get rid of non-Euclidean geometry by preserving gravitation as a real force. Einstein's reference to Poincaré in *Geometrie und Erfahrung* (Einstein 1921a) probably played some in Reichenbach's conversion. However, the logical empiricists failed to notice (see Schlick's commetary in Helmholtz 1921; Schlick 1921; Reichenbach 1921, 355; 1878, I, 33) that Einstein's reference should be read in the context of Einstein's "measuring rod objection" against Weyl's theory of electromagnetism (Weyl 1918a; Einstein 1918b), rather than as an argument for geometrical conventionalism (Ryckman 2005, § 3.5; see also Giovanelli 2012).

In a paper published in French in 1922 (Reichenbach 1922, tr. 2006), Reichenbach explicitly claims that "[t]he solution to the problem of space is therefore found only in this conception we call conventionalism which goes back to Helmholtz and Poincaré" (Reichenbach 1922, 40; tr. 2006, 135). If the measurements with rigid rods yield a non-Euclidean geometry, one could, alternatively, maintain that the geometry of space-time was Euclidean, holding that measuring instruments are actually non-rigid, and deformed by a non-detectable force of type X, which causes uniform shrinkages and expansions in all materials. According to Reichenbach, "the real problem lies in deciding between these alternatives: either Euclidean geometry and a field X or the geometry determined by experience and no field X" (Reichenbach 1922, 40; tr. 2006; 135). The empirical facts can force us to select between either the Euclidean or the non-Euclidean description as the uniquely correct description.

Of course, gravitation, in Reichenbach's terminology is "a force of type X," (Reichenbach 1922, 41; tr. 2006; 132); as a consequence of the identity of inertial and gravitational mass, it affects all bodies in the same way. Einstein's choice "to set $X = 0$" (Reichenbach 1922, 43; tr. 2006; 129) and abandon Euclidean geometry was the simplest choice, but not the only one possible. According to Reichenbach, the very existence of these alternatives represents the "characteristic of the epistemological solutions for which we are indebted to the theory of relativity" (Reichenbach, 1922, 40; tr. 2006; 135).

When, however, Reichenbach describes in some detail why gravitation is a force of type X, it is not the choice among different geometries that comes to the fore:

> It is also in this fashion that we can obtain the remarkable identification of gravitation with certain fictitious forces that result from a change in coordinates. Imagine a gravitation-free space in which we find a system of coordinates formed by a network of congruent rigid rods. In this space the metric is Euclidean; that is to say the $g_{\mu\nu}$ possess the well-known special form $[ds^2 = \sum dx_\nu^2]$. If we *now introduce new coordinates* such that the rods get progressively shorter the farther to the exterior a rod is, so that the network will be considered to be curved, then the new $g_{\mu\nu}$ take a form different from the special form. This can be conceived of in the following way: *there exists a force which shrinks the rods* and this force is represented by the deviation of the $g_{\mu\nu}$ from the special form; it is therefore considered to be a correction in the establishment of the ds^2. It is clear that this force … is only a fictitious force produced by the anomaly of the rods. All of the magnitudes that, in an element of the network, are measured with a local unit *undergo the same correction*; this is precisely the motivation for considering this fictitious force to be interpreted as a gravitational field (Reichenbach 1922, 39; tr. 2006; 134; my emphasis).

In the example considered by Reichenbach, one is not in front of alternative geometries; it is the very same flat geometry in different coordinate systems. A matter-free region of space-time appears to be devoid of gravitation from the perspective of one coordinate system and endowed with a gravitational field when considered from another coordinate system: the difference is one of description. Gravitation is a "fictitious force" because it is coordinate-dependent: as Reichenbach's writes "the gravitational field and the corrections resulting from*a simple change in coordinates* can be brought together in *a single concept*" (Reichenbach 1922, 39; tr. 2006; 134).

This is of course the very core of the equivalence principle as Einstein originally presented it. As Reichenbach explains, although only in a brief footnote, "it is necessary here to draw a distinction between the gravitational potential and the gravitational field", that is, between *"the gravitational potentials* (the $g_{\mu\nu}$)" and *"[i]ts gradient, the field*" (Reichenbach 1922, 39, n.; tr. 2006; 134, n.). The components of the metric represents the "gravitational potentials", whereas, like in other field theories, the "gradient of the potentials" is the natural candidate for representing the gravitational field.

What distinguishes the gravitational field from other fields is the fact that in a flat region of space-time, non-constant potentials $g_{\mu\nu}$ can be introduced by "a simple change in coordinates". The transition to the general theory of relativity is realized by the assumption that such a representation of the field by the non-constancy of the functions $g_{\mu\nu}$, is also justified in the general case in which the metric cannot be reduced to quasi-Euclidean form of the special theory of relativity by "a simple change in coordinates".

In his more technical writing *Axiomatik der relativistischen Raum–Zeit–Lehre* (Reichenbach 1924), Reichenbach emphasizes that what characterizes a local inertial system is the fact that the gradient of the metric $\frac{\partial g_{\mu\nu}}{\partial x_\sigma}$ can be made to vanish by a coordinate transformation, whereas in a general gravitational field this is impossible, as the gradient of the metric $\frac{\partial^2 g_{\mu\nu}}{\partial x_\sigma \partial x_\rho}$ does not vanish. However Reichenbach's philosophical conclusion is stunningly different (cf. Reichenbach, 1924, 106; tr. 1965, 133).

Focusing on what he now calls "metrical forces" (Reichenbach, 1924, 68; tr. 1965, 87) (because they depend on the choice of the metric), Reichenbach is convinced that "space and time in the general theory of relativity mean the same as in the special theory *although without any metric*" (Reichenbach 1924, 155; tr. 1965, 195; my emphasis); the "topological properties turn out to be more constant that the metrical one", so that Reichenbach famously argues that "the transition from the special theory to the general one represents merely a *renunciation of metrical characteristics*, while the fundamental *topological character* of space and time remains the same." (Reichenbach 1924, 155; tr. 1965, 195)

In the immediately subsequent years, logical empiricists quickly came to agree on the fact that such an opposition between the topological and the metrical properties of space-time is the relevant innovation introduced by general relativity. Carnap, in his first post-doctoral writings (Carnap 1923, 1925), could easily translate his early "Kantian" conventionalism into an empiricist framework (Carnap 1922). Only topological space reproduces what is present in experience uni-vocally. By contrast, all post-topological structure depends upon a stipulation.

5 Reichenbach's Mature Conventionalism and the "Analytic Treatment of Riemannian Spaces"

By the end of 1926 [4], Reichenbach had already finished his semi-popular *Philosophie der Raum–Zeit–Lehre* (1926, but published only as Reichenbach 1928; tr. Reichenbach 1958), which furnishes a very elegant and effective presentation of this doctrine and of the noble tradition from which it emerges:

> This conception of the problem of geometry is essentially the result of the work of Riemann, Helmholtz, and Poincaré and is known as conventionalism. While Riemann prepared the way for an application of geometry to physical reality by his mathematical formulation the concept of space, Helmholtz laid the philosophical foundations. In particular, he recognized the connection of the problem of geometry with that of rigid bodies … It is Einstein's achievement to have applied the theory of the relativity of geometry to physics. The surprising result was the fact that the world is non-Euclidean, as the theorists of relativity are wont to say; in our language this means: if $F = 0$, the geometry G becomes non-Euclidean. This outcome had not been anticipated, and Helmholtz and Poincaré still believed that the geometry obtained could not be proved to be different from Euclidean geometry. Only Einstein's theory of gravitation predicted the non-Euclidean result which was confirmed by astronomical observations (Reichenbach 1928, 48; tr. 1958, 35).

[4] cf. Reichenbach's letter to Schlick on December, 6 1926 mentioned in (Schlick 2006, vol. 6, 175)

According to what Reichenbach calls theorem θ, a non-Euclidean geometry G is equivalent to an Euclidean geometry G' with a field of "universal forces" F. Only the combination $G + F$ is testable by experience, after a conventional choice has been made. Einstein chose the simplest convention by setting $F = 0$.

The action of such a force is not completely arbitrary. In order to avoid causal anomalies (Reichenbach 1928, § 12), the original geometry G must be mapped into the new one G' uniquely and continuously. The very lesson we can draw from Einstein's theory is that we are *free to choose* among *topologically equivalent*, but *metrically different spaces*, that can be smoothly deformed into one another, by some "force" F that affects all bodes. The "metrical relations are distorted" (Reichenbach 1928, 48; tr. 1958, 35), whereas the "topological" structure remains untouched. According to Reichenbach, the world as it is in-itself does not have a unique metric; it does, however, have a unique topology, and this is defined in terms of the causal relations.

Reichenbach believed that Riemann had established the mathematical framework for this philosophical conclusion. Riemann's "mathematical achievement" is "of greatest significance for the epistemological problem of space" (Reichenbach 1928, 244; tr. 1958, 279f.). "The mathematical treatment" divides the description of a space "into a topological and a metrical part": "the function of numbering" is assigned to the coordinate system on the one hand; whereas "the metrical functions of measuring lengths" are assigned "to the metrical coefficients $g_{\mu\nu}$" (Reichenbach 1928, 244; tr. 1958, 280) on the other. In Riemannian geometry the "metrical function of the $g_{\mu\nu}$ plays a *subordinate role*. It cannot change the *topological foundation* determined by the coordinate system. It merely adds to it a *metrical superstructure* [Überbau]" (Reichenbach 1928, 244; tr. 1958, 280).

5.1 Reichenbach's Technical Presentation of Riemannian Geometry and its Incompatibility With His Own Conventionalism

In Reichenbach's view Riemann deprived the coordinate system of all but topological properties. This interpretation seems however to be based on a fundamental misunderstanding. Indeed, in Riemannian geometry coordinates have no metrical significance *per se*; they are only a set of markers that serve to distinguish the points. Yet such a significance is regained by the introduction of a quadratic differential form with variable coefficients $g_{\mu\nu}$ as function of the coordinates. Instead of separating the metric and topological significance of the coordinate system, Riemannian geometry shows that there is no other source of information about the coordinates apart from the $g_{\mu\nu}$.

One is free to introduce any coordination of the physical space that is produced by an arbitrary, smooth transformation from the original. However, this transformation is accompanied by a suitable change of the $g_{\mu\nu}$, so that that the ds^2 are unchanged; all measured relations can be "recovered" in the new coordinate system by using the new $g'_{\mu\nu}$ to get real distances from coordinate distances. "The numbers $g_{\mu\nu}$ indicate how, at a given place, the length of the line element is to be calculated from the coordinate differentials" (Reichenbach, 1928, 243; tr. 1958, 279).

As Reichenbach has explained in his 1920 monograph, the coordinate system is arbitrary, but the dependency of the $g_{\mu\nu}$ from the coordinate system has an objective meaning. This still emerges clearly from Reichenbach's own presentation of the "Analytic Treatment of Riemannian Spaces" in his 1926/1928 book:

> This fact is expressed analytically by a property of the $g_{\mu\nu}$, as we shall see when we investigate transformations to other coordinate systems. Let us imagine that a second coordinate system has been introduced and that the position of the new family of lines is given as a function of the old coordinates, … We now add *the restriction* that the transition to the new coordinates *must not change any of the metrical relations*; this transition, therefore, leads only to a *different form of description*. We must then specify a new system $g'_{\mu\nu}$ of metrical coefficients relative to the new coordinates x'_{μ} such that the old relations of congruence are preserved. If two line segments are equally long when measured by the old $g_{\mu\nu}$ in the old coordinate system, they must still be equally long when measured by the new $g'_{\mu\nu}$ in the new coordinate system. This requirement leads to the condition that
>
> $$ds^2 = g_{\mu\nu}dx_{\mu}dx_{\nu} = g'_{\sigma\tau}dx'_{\sigma}dx'_{\tau}$$

We can therefore say that ds^2 is an invariant of the transformation, and we can show from (8) how the new $g_{\mu\nu}$ are to be calculated from the old ones. (Reichenbach 1928, 281; tr. 1958, 245; my emphasis)

As we have seen, a classic example is the passage from using Cartesian (rectangular) coordinates to polar coordinates. The points are relabeled with different numbers (for instance, in a surface the point with Cartesian coordinates (1, 1) has polar coordinates $\sqrt{2}, \frac{\pi}{4}$). However, regardless of the coordinate system used, the distances between any two pairs of points are assigned the *same value*. The same flat Euclidean metric is expressed in a different coordinate system.

Geometrical conventionalism violated exactly this condition, which in Riemann's approach should be considered as fundamental. This emerges clearly from Reichenbach's attempt to cast the loose language of universal forces in a more formal framework: if the results of measurement yield a metric $g_{\mu\nu}$ which is non-Euclidean, then one can infer that the geometry is actually given by the normal matrix $\overline{g_{\mu\nu}}$, if one stipulates that our measuring rods were under the influence of a universal force F.

> Generally the force F is a tensor. If the $g'_{\mu\nu}$ are the metrical coefficients of the geometry G' and $g_{\mu\nu}$ those of G, the potentials $F_{\mu\nu}$ of the force F are given by $g'_{\mu\nu} + F_{\mu\nu} = g_{\mu\nu}$. The measuring rods furnish directly the $g'_{\mu\nu}$; the $F_{\mu\nu}$ are the "correction factors" by which the $g'_{\mu\nu}$, are corrected so that $g_{\mu\nu}$ results. The universal force F influencing the measuring rod is usually dependent on the orientation of the measuring rod (Reichenbach 1928, 44, n.; tr. 1958, 33).

Let the $g'_{\mu\nu}$ represent some non-flat geometry; then the introduction of the arbitrary potentials $F_{\mu\nu}$ of a universal force field leads us to conclude that the metric of the space can be reduced to a Euclidean flat one $g_{\mu\nu}$: $ds^2 = g_{\mu\nu}dx_\mu dx_\nu = (g'_{\mu\nu} + F_{\mu\nu})dx'_\mu dx'_\nu$ (Norton 1994). The equation deals precisely with a $g_{\mu\nu}$ system which *cannot* be transformed into $g'_{\mu\nu}$ by a coordinate transformation if the line element has to be preserved. In fact, if this equation is valid, then trivially the quadratic form is not an invariant $ds^2 = g_{\mu\nu}dx_\mu dx_\nu \neq g'_{\mu\nu}dx'_\mu dx'_\nu$. If two tracts are congruent when measured by the old $g_{\mu\nu}$ in the old coordinate system, then *they are not* equally long when measured by the new $g'_{\mu\nu}$ in the new coordinate system. Thus, the two geometries can be transformed into one another only if one renounces the invariant character of ds^2; in other words, one deprives space of its metrical properties, i.e. the structure which tells how much space or time lies between point-events.

5.2 Reichenbach on Riemann's *Äquivalenzproblem*

There is a sort of conflict between Reichenbach's philosophical intentions and Reichenbach's own popularization of the mathematical apparatus of Riemannian geometry. Philosophically, Reichenbach attempted to interpret Riemann's approach from the perspective of Helmholtz's and Poincaré's problem of finding a criterion for choosing between *different metrical geometries* that only agree on the "topological order of all space-points determined through the coordinate system" (Reichenbach 1929b, 683). Riemann's main concern, in contrast, was that of finding a criterion to discern the properties of the *same metrical geometry* that depend on the choice of a coordinate system from those that are coordinate independent. This can be understood only if one considers Riemann's geometrical insight from the perspective of the non-geometrical development put forward by Christoffel and Ricci.

From this vantage point, as again Reichenbach explains very clearly in the less philosophical parts of the book, it appears that Riemann wanted to investigate different*classes of quadratic differential forms*. Each class, insofar as the line elements can be transformed into one another by simple coordinate transformation, represents the *same geometry*. On the contrary, $g_{\mu\nu}$-systems, which cannot be transformed into one another by any change of the coordinates represents *different geometries*:

> We may restate our results as follows: any given system $g_{\mu\nu}$ can be transformed into another system $g_{\sigma\tau}$, by means of [a smooth change of variables]. Transformations of this kind, starting with a definite set $g_{\mu\nu}$ do not give us all conceivable systems, however, but merely *a limited class*. The systems of this class are *geometrically equivalent* to the initial $g_{\mu\nu}$, and the class as a whole characterizes *a definite geometry*. Other classes similarly constructed, would characterize *another geometry*. A special class is the class which contains the normal system; it is the class of Euclidean geometry (Reichenbach 1928, 282f.; tr. 1958, 246).

The very purpose of Riemann's investigation was to decide when two $g_{\mu\nu}$-systems differ only by a coordinate transformation and are therefore *geometrically* equivalent. Two metrics are equivalent, if and only if there is a coordinate transformation that transforms $g_{\mu\nu}$ into $g'_{\mu\nu}$ so that $ds^2 = ds'^2$. This is of course precisely the *Äquivalenzproblem*. As we have seen, it is mainly due to the merit of Christoffel and Ricci that the mathematical technique for dealing with this problem was developed. Reichenbach describes it once again very accurately:

> The question now arises whether there exists a special characteristic of the class of Euclidean Geometry. Mathematicians have shown that it is possible to formulate such a criterion. For this purpose one has to form a certain mathematical combination of the $g_{\mu\nu}$ and $g_{\mu\nu}, \frac{\partial g_{\mu\nu}}{\partial x_\sigma}, \frac{\partial^2 g_{\mu\nu}}{\partial x_\tau \partial x_\sigma}$ which is called $R_{\mu\nu\sigma\tau}$: ... $R_{\mu\nu\sigma\tau}$ is therefore a tensor of rank 4. We can recognize directly … a very important property of all tensors, namely, that if all components of a tensor are zero in one coordinate system, they will all be zero in every other coordinate system. … Since it can be shown that $R_{\mu\nu\sigma\tau}$, vanishes for the normal system, it follows that every system of the Euclidean class is characterized by the condition
>
> $$R_{\mu\nu\sigma\tau} = 0$$

$R_{\mu\nu\sigma\tau}$ is called the Riemannian curvature tensor. It is a measure of curvature. (Reichenbach 1928, 283; tr. 1958, 246)

As we have seen, $R_{\mu\nu\sigma\tau}$ entails the first and second derivatives of the $g_{\mu\nu}$. Since it is a tensor, if it vanishes in one coordinate system, it vanishes in all coordinate systems. All inter-transformable $g_{\mu\nu}$-systems represent the same Euclidean geometry, since in all these cases, even if the $g_{\mu\nu}$ can be variable, the Riemann tensor vanishes. On the contrary, the Christoffel symbols $\Gamma^{\tau}_{\mu\nu}$, which entail the first derivatives of the $g_{\mu\nu}$, are not tensors. If they vanish in one coordinate system, they do not vanish in another. Hence we cannot expect that the $g_{\mu\nu}$, or the $\Gamma^{\tau}_{\mu\nu}$, should give something of absolute significance. We have to differentiate the $g_{\mu\nu}$ twice before we arrive at something that has a significance independent of any special coordinate system.

6 Relativity of Geometry Versus Relativity of Gravitation

Reichenbach was philosophically convinced the that "metrical properties of the space-time continuum are *destroyed by gravitational fields*" (Reichenbach 1928, 269; tr. 1958, 308; my emphasis); since an alternative to Einstein's stipulation would have been in principle possible, space-time has no definite metrical structure. However, once again, following Reichenbach's semi-technical presentation of general relativity rather than his philosophical interpretation, one finds a more humble truth: the gravitational fields, merely "*destroy the orthogonal form of the line element*" (Reichenbach 1928, 289; tr. 1958, 253; my emphasis).

Let us consider again an inertial frame K without a gravitational field, that is, a frame where "the $g'_{\sigma\tau}$ will satisfy the normal matrix. If we now describe the same local world region from an accelerated system K, the $g_{\mu\nu}$ of this system can *no longer satisfy the normal matrix*"; hence "the $g_{\mu\nu}$ will characterize the acceleration of *K*" (Reichenbach 1928, 288; tr. 1958, 253); but, because of the equivalence principle "[i]f they characterize the state of acceleration of *K*, " (Reichenbach 1928, 289; tr. 1958, 253), "they must also characterize the gravitational field which exist for *K*". Of course, the geometry of space has not changed in passing from $g'_{\sigma\tau}$ to $g_{\mu\nu}$; rather, it is the same flat Minkowski geometry, represented in different coordinate systems (Reichenbach 1928, 289; tr. 1958, 253).

6.1 The Equivalence Principle and the Christoffel-Symbols

As Reichenbach already briefly explained in a footnote in his 1922 article, Einstein identified the gravitational field with *the gradient of the potentials*$g_{\mu\nu}$ that is, with the non-vanishing of the Christoffel symbols $\Gamma^{\tau}_{\mu\nu}$. As in every field theory, in gravitational theory one distinguishes the "concepts of potential and gradient"; in electricity theory, the force field is the gradient of the electric potentials; analogously, in general relativity "the gravitational force will … be characterized by the *potential gradient* [Potentialgefälle] which can be calculated for every point from the *potential field*" (Reichenbach 1928, 268; tr. 1958, 233).

According to Reichenbach "[t]his representation explains why the gravitational field can be transformed away" (Reichenbach 1928, 271; tr. 1958, 236). As we have seen, in Einstein's original approach, within a small enough region of space-time, one can introduce an "artificial gravitational" field by simply introducing a coordinate system, in which the $g_{\mu\nu}$ are not constant, but become functions of the coordinates. For the very same reason one can make this gravitational field disappear through a coordinate transformation: we can set "the metrical field in such a manner that the components, the gravitational potentials [$g_{\mu\nu}$] become constants (this is always possible at least for local regions); then there exists no gravitational *gradient*. The *disappearance of the gradient is then called 'the disappearance of the gravitational field.'*" (Reichenbach, 1928, 271; tr. 1958, 236; second emphasis mine).

Reichenbach suggested then that we should distinguish between the tensor as $g_{\mu\nu}$ "as a whole or the metrical field", "the particular sets of tensor components", in a certain coordinate system and "finally the particular set of gradient coefficients of the tensor components" (Reichenbach 1928, 271; tr. 1958, 236):

> In the mathematical representation, the metrical field is given by the tensor $g_{\mu\nu}$ the gravitational potential field by the particular set of components $g_{\mu\nu}$, and the gravitational gradient field through the Riemann–Christoffel symbols $\Gamma^{\tau}_{\mu\nu}$, which are obtained from the $\frac{\partial g_{\mu\nu}}{\partial x_{\tau}}$. The $\Gamma^{\tau}_{\mu\nu}$ do not form a real tensor, only a linear tensor, and can therefore all at once be transformed to zero by nonlinear transformations. A fourth concept has occasionally been introduced. We set $\overline{g_{\mu\nu}} + \gamma_{\mu\nu}$, where $\overline{g_{\mu\nu}}$ are the normal orthogonal values of the $g_{\mu\nu}$, and we refer to the $\overline{g_{\mu\nu}}$ as the *inertial field* and only to the $\gamma_{\mu\nu}$ as the gravitational potential field. The $\Gamma^{\tau}_{\mu\nu}$ may then be considered as the derivatives of the $\gamma_{\mu\nu}$, since the $\overline{g_{\mu\nu}}$ as constants do not contribute to the gradient field. This resolution into inertial and gravitational field is an adaptation to the terminology

of Newtonian mechanics, however, and is therefore hardly appropriate (Reichenbach 1928, 272; tr. 1958, 237).

This passage is curiously written in smaller characters; Reichenbach's intention is simply to introduce some mathematical technicalities. This is revealing of Reichenbach's philosophical attitude. From today's point of view, the passage just quoted in fact seems to describe the very conceptual core of Einstein's path to the theory of general relativity. The conceptual difference between gravitational "fields" and the other "fields" is that the latter, according to Einstein, should be represented by a non-tensorial, coordinate-dependent quantity, the Christoffel symbols $\Gamma^{\tau}_{\mu\nu}$. Locally, the phenomena of gravity and acceleration were, in Einstein's view, two ways of looking at the *same space-time* in terms of different coordinate systems.

Reichenbach is so aware of the importance of this point that he even addresses the problems which arise following Einstein's original approach: "If we change to three-dimensional polar coordinates, for example, while the time coordinate remains unchanged, the $g_{\mu\nu}$ will assume *a non-standard from*. For these coordinates there must therefore exist a gravitational field", since in polar coordinates "the partial derivatives [$\frac{\partial g_{\mu\nu}}{\partial x_\tau}$] do not vanish throughout" (Reichenbach 1928, 290, n.; tr. 1958, 253). However, this is rather counterintuitive since no change in the state of motion is provided by a mere spatial transformation, which leaves the g_{44} unchanged (see for instance Reichenbächer 1923). This "says more than was originally expressed by the principle of equivalence" (Reichenbach 1928, 289; tr. 1958, 253).

It would therefore be advantageous to express the gravitational field in a coordinate invariant form (on this point see Eddington, 1923, 39f.): "all $g_{\mu\nu}$-systems derived from a $g'_{\sigma\tau}$-system by means of coordinate transformations are merely different resolutions of *the same tensor into different sets of components*. This tensor, the metrical field, is therefore independent of specific coordinate systems" (Reichenbach 1928, 289; tr. 1958, 253). A physical magnitude expressed by a tensor has definite components once a basis is given in a chosen coordinate system, but abstractly considered, it stands for its components in all coordinate systems. All transformable systems should represent the same gravitational field, the presence of which should be better expressed by the non-vanishing of Riemann–Christoffel tensor $R_{\mu\nu\sigma\tau}$. However, this would mean accepting that "the consequence that transformations of the state of motion will not change the gravitational field either, since they too leave the *metrical field invariant*" (Reichenbach 1928, 291; tr. 1958, 254). In this way the "principle of equivalence" (that is, the possibility of interpreting locally an acceleration field as gravitational field) would become useless.

6.2 Co-Variant and Invariant

As Reichenbach notices in the popular book *Von Kopernikus bis Einstein. Der Wandel unseres Weltbildes* (Reichenbach 1927), all of this can be understood precisely because Einstein "had to introduce in physics a new mathematical method, the so-called tensor calculus". "The essence of the new method of calculation resides in two basic concepts, the invariant and the co-variant" (Reichenbach 1927, 105). In particular Reichenbach suggests interpreting the *gravitational field as a covariant magnitude*, which depends on the coordinate system, and*the metric field as an invariant magnitude*:

> This consideration leads to a distinction which we have touched upon several times before and which expresses *a basic idea of modern science*. The system of the tensor components is covariant, i.e. it has a *different numerical composition* for each coordinate system. Yet we express in this fashion a state that is independent of the coordinate system, i.e., an *invariant state*. The tensor as a whole is an invariant magnitude. We can recognize this property from its representation by means of components, since the components can be calculated for every coordinate system, if they are known for one. It is unfortunate that the physical terminology does not reflect this well-defined mathematical distinction. By *"gravitational field*" we understand the system of components of the tensor in each case; this makes the gravitational field a covariant magnitude. No particular term has been accepted for the invariant tensor field as a whole. It might best be called the *metrical field*, in accordance with some ideas which we shall discuss later; in fact, this term has occasionally been used with this meaning. In this terminology the gravitational field is the particular system of components into which the metrical field has been resolved (Reichenbach 1928, 271; tr. 1958, 236).

Reichenbach's terminology is surely non-standard. As we have seen, Reichenbach himself notices that the gravitational field is represented by the Christoffel symbols, that is, by a non-tensorial, i.e. non-covariant quantity (cf. above p. 12). However, the point Reichenbach wants to make is clear enough: The values of the components $g_{\mu\nu}$ depend on a particular coordinate system, they are *covariant quantities*. However, regardless of the coordinate system used, the lengths of lines are assigned the same value, that is, the length between a pair of space-time points is *an invariant quantity*.

According to Reichenbach, "[t]he coordinate systems themselves are not equivalent", in the sense that in every system a different set of tensor components $g_{\mu\nu}$ is defined; however, "every coordinate system with *its* corresponding gravitational field is equivalent to any other coordinate system together with its corresponding gravitational field" (Reichenbach 1928, 272; tr. 1958, 237). The ensemble of all inter-transformable $g_{\mu\nu}$-systems represent the same metric field and thus the same geometry of space-time: "Each of these covariant descriptions is an admissible representation of the *invariant state* of the world" (Reichenbach 1928, 272; tr. 1958, 237). Similarly, an electromagnetic field as a whole transforms as a tensor, but not the electric and the magnetic field separately: an electric field for one observer could be a superposition of an electric and a magnetic field for the other.

Surprisingly, from Reichenbach's semi-technical presentation of the general theory of relativity one learns that "*the gravitational field is deprived of its absolute character* and recognized as a covariant magnitude" (Reichenbach 1928, 248; tr. 1958, 214), that is, (in Reichenbach's parlance) a coordinate-dependent magnitude, represented by $\Gamma^{\tau}_{\mu\nu}$. In a flat Minkowski space-time, an "artificial" gravitational field can be introduced by a mere coordinate transformation. On the other hand, the *geometry of space-time has an absolute character* in the sense that it is coordinate independent, it does not change if one represents it in different coordinate systems; for instance, Minkowski space-time is flat, which means that the Riemann tensor $R_{\mu\nu\sigma\tau}$ vanishes everywhere regardless of the coordinate system used.

Following the more expository parts of Reichenbach's book one discovers that there is indeed *a conventional element* in the general theory of relativity; however, this is not the geometry, but rather the *gravitational field*. The same space-time geometry, which in one coordinate system has only an inertial component, in another it has both an inertial and a gravitational component. Since the Christoffel-symbols are not tensors, if they have zero components, that is, only inertial components in one coordinate system, there exists a coordinate system in which the components are non-zero and a gravitational field appears.

The consequence is that the distinction between inertia and gravitation is a *mathematical artifact* which depends on the choice of the coordinates and therefore does not reflect a real physical difference: only the sum of the two pieces represents something physical. Following this line of thought, which has somehow remained hidden behind the curtains of Reichenbach's "official" philosophy, one discovers that the core of Einstein's original approach should not be found in the discovery of the *relativity of geometry*, but rather in the discovery of *relativity of the gravitational field* (Janssen 2005).

6.3 "Displacement Space" Versus "Metrical Space"

From this point of view, Reichenbach's idea that the gravitational force has to be set equal to zero is at least ambiguous (Torretti, 1983, 237). The best way to understand the fact that gravity is a "universal force" is precisely to recognize that the gravitational field is not a tensor field on the manifold, whereas the potential field is.

This point, however, gets completely lost using the conceptual resources of the Helmholtz and Poincaré debate on the foundation of geometry. The non-geometrical work of Christoffel and Ricci furnished Einstein the analytical tools to express this distinctive feature of the gravitational field. As we have mentioned (p. 3.3), a geometrical reinterpretation of this mathematical apparatus was developed only after general relativity, in particular through Levi-Civita's notion of parallel displacement of a vector.

Reichenbach, however, always considered the discovery of "the independence of the displacement operation [Verschiebungsoperation] … given by the $\Gamma^{\tau}_{\mu\nu}$ from the metric … given by the $g_{\mu\nu}$" (Reichenbach 1929b, 683) as philosophically irrelevant. The quantities $\Gamma^{\tau}_{\mu\nu}$ cannot be measured directly by rods and clocks, but must be obtained from the directly measured quantities $g_{\mu\nu}$ by calculation: they are a "product of fantasy, mere illustration" (Reichenbach, 1928, 352; on this point see Coffa 1979). Also, much later works of Reichenbach's (Reichenbach 1949; Reichenbach 1951) do not show any sign of "resipiscence".

From today's point of view, however, lurking here is the crucial point of Einstein's original approach. Contrary to modern relativists (Synge 1960, VIIIf.;

Synge 1970, 158; Ohanian 1977; see Lehmkuhl 2008), Einstein insisted until the end of his life that "what characterizes the existence of a gravitational field … is the non-vanishing of the Γ^l_{ik}, not the non-vanishing of the R_{iklm}" (Einstein to Max von Laue, September 1950; translated in Stachel 1986, 1858). As he points out in one of his last letters to Michele Besso in 1954, "[t]his was still not so clear at the time of the setting up of the G.R., but was subsequently recognized principally through Levi- Civita", who "provided the possibility of defining the 'displacement field' Γ^l_{ik}", which "is in-itself independent from the existence of a metric field g_{ik}" (Einstein 1972, 525–527; partially translated in Norton 2002, see also Einstein 1956, 141).

The vanishing of Christoffel symbols does not mean that there is no gravitational/displacement field (Giulini 2001, § 9). A gravitational field can always be introduced, even in a flat Minkowski space-time, by a simple change in coordinates. Thus, the flat metric of space-time can be regarded as a special case of a gravitational field, rather than the absence of a gravitational field (see also Einstein to Becquerol, August 16, 1951, cited by Norton 1985). This seems to be in harmony with Einstein's famous claim that there is no "space without a field". If we imagine the gravitational field to be removed, there remains "absolutely *nothing*, and also no 'topological space"' (Einstein 1952, 155).

7 Conclusion. Over-Determination Versus Under-Determination

Reichenbach's idea that "the topology of space is to be regarded as a more fundamental determination than the metric" (Reichenbach, 1929a, 30; tr. 1978, I, 183) seems to derive from the observation that the metric relations, the lengths of worldlines, *do not* remain invariant under an arbitrary deformation of space-time (induced for instance by a universal force). Such a deformation would preserve only the smoothness of the coordinate system and the uniqueness of the labeling of the points, depriving the coordinate system of all but its topological properties.

However, distances *do* remain invariant under a transformation which represents a mere recoordinization, that is, under the condition that the coefficients of the quadratic form also change. This is exactly the relevant point. The geometry of a given space-time is characterized by the invariant interval between any two world-points; once a unit of measure has be chosen, the numerical value of this interval remains unchanged under coordinate transformations.

What is relevant in general relativity, as Reichenbach showed in his 1920 book, is precisely the "relativity of coordinates" and not the "relativity of geometry" that appears in his mature philosophy: "The *relativity of geometry* is a consequence of the fact that *different geometries* can be represented by one another with a one-to-one correspondence" (Reichenbach 1949, 298). The *relativity of coordinates* is due to the fact that the *same geometry* can be represented in different coordinate systems.

As we have tried to show, the origin of Reichenbach's "conversion" can be regarded as the result of a "collision" of mathematical traditions (analogous, although not perfectly identical, to that denounced in Norton 1999). Instead of *following* the Riemann–Christoffel–Ricci–Einstein line, as he seemed to do in his first monograph, Reichenbach, probably under the influence of Schlick, tried to *create* the tradition Riemann-Helmholtz-Poincaré-Einstein:

> The solution to the problem of space described here is to be attributed principally to the work of Riemann, Helmholtz, Poincaré, and Einstein, Helmholtz, the first to acknowledge the significance of Riemann's idea for physics, indisputably deserves the major credit for the recognition of the definitional character of congruence in physical space. Poincaré coined the term conventionalism, which refers to the definitional character of the congruence of line segments and designated the definition in question as a convention. At the time he introduced this idea, Poincaré still believed that the convention of the rigid body led to Euclidean geometry, not knowing that Einstein was soon to take up the idea of conventionalism in all seriousness and apply non-Euclidean geometry to physics, final clarification came about with the philosophical discussion of Einstein's general theory of relativity (Reichenbach 1929a, 60; tr. Reichenbach, 1978, 179)

As recent literature has abundantly shown, general relativity cannot be the heir of this philosophical/mathematical tradition simply because such a tradition does not exist. Poincaré explicitly excluded the Riemanniann geometries of variable curvature from his conventionalism, precisely because they were at odds with the Helmholtzian approach based on the existence of rigid bodies. As we have tried to point out, Riemann's geometrical approach to geometry actually remained dormant and developed instead in a non-geometrical tradition, the main protagonists of which are arguably Christoffel and Ricci. The geometrical implications of the calculus were developed mainly after the emergence of general relativity by Levi-Civita and others.

A much more plausible line of development of Riemann's work in the 19th century is, for instance, that which was suggested by Cartan in a non-technical paper of 1931:

> Euclidean geometry itself also uses analytic methods relying on the use of coordinates, but these coordinates (Cartesian, rectangular, polar, etc.) have a precise, quantitative geometric significance, which is why they can be introduced only after the geometry is founded by its own methods. In Riemannian geometry, on the contrary, coordinates, introduced from the beginning, serve simply to relate empirically the different points of space, and geometry has precisely the object of extricating the geometric properties that are independent of this arbitrary choice of coordinates. … The necessity of using systems of arbitrary coordinates exerted a profound influence on the later development of mathematics and physics. It led to the admirable creation of an absolute differential calculus by Ricci and Levi-Civita, which was the instrument that helped to elaborate general relativity (Cartan, 1931, 397f.; tr. Pesic 2007, 182).

We are dealing with two mathematical traditions that seem to be interested in different philosophical problems. Helmholtz and Poincaré tried to determine the condition under which it is possible to make a choice among *alternative physical* *geometries* that share some weaker mathematical structure. In contrast, Riemann raised the problem of distinguishing the *intrinsic features of a given physical geometry* from the apparent differences introduced by the choice of a particular coordinate system.

There is a broader philosophical lesson we can take away from the historical reconstruction outlined above. Considered in the light of the 19th century philosophy of geometry, the logical empiricists could interpret general relativity as a case of *mathematical undertermination*: the relevant geometric structure that General Relativity has at its disposal (what the Logical Empiricist call the "topological" structure) is not "rich enough" to allow a choice among different possible physical geometries. Considered under the historical perspective we have suggested, the requirement of general covariance, if it is not to be considered trivial, seems to raise instead a problem of *mathematical overdetermination*; that is, the physical system is described by a surplus of mathematical structure (Norton 2003; more in general on this topic Redhead, 2001).

The logical empiricists' strategy was to introduce some *redundant physical elements* (the universal forces) in order to extract the *mathematically relevant* content of the theory. From a contemporary perspective, on the contrary, general relativity seems instead to introduce *redundant mathematical structure* in order to extract the *physically relevant* content of the theory as the invariant content (Giulini 2007).

Einstein's notorious "hole argument", Hilbert's version of the argument in terms of Cauchy initial conditions (Stachel 1992), are precisely the consequence of the existence of mathematical degrees of freedom that do not have any correspondence in physical reality. The covariance of Einstein's equations leaves undetermined the evolution of four out of the ten components of $g_{\mu\nu}$. Instead of being a trivial consequence of the application of the "absolute differential calculus" as in the "Kretschmann objection", such a "redundancy" in mathematical formalism appears rather to be one of the main philosophical issues raised by general relativity (Norton, 2003).

Historically, this problem was "rediscovered" in the late 1950's by Peter Bergmann (Bergmann 1956; Bergmann 1961b; Bergmann and Komar 1960; Bergmann 1961a), Einstein's assistant at Princeton, who was led to again discuss the question of what is observable in general relativity. From this vantage point, the Logical Empiricists' idea that the metric is not an "observable", whereas the neighborhood relations encoded in the coordinate system are, appears then to be utterly inadequate in grasping the relevant philosophical issue; general relativity shows instead that, because of "the mathematical ambiguity of the coordinate system" (Bergmann 1956, 491), the values of the metric *at a particular world-point* are not an observable (Bergmann 1961b, § 27; Bergmann, 1968, § 27).

While Reichenbach continued to insist in his last writings that the main philosophical issue of Einstein's general theory was the existence of a "class of equivalent descriptions" (Reichenbach, 1951, 133) of *different physical situations*, after Reichenbach's death it became clear that the real problem in General Relativity is that, as Bergmann put it, there is an "equivalence class of solutions" that describes the "*same physical situation*" (Bergmann 1961b, 511). Only the equivalence class is physically real.

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